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What Makes Mathematics Manipulatives Effective? Lessons From Cognitive Science and Montessori Education

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Abstract

Manipulatives are ubiquitous in early childhood classrooms; yet, findings regarding their efficacy for learning mathematics concepts are inconsistent. In this article, we present four general principles that have emerged from cognitive science about ways to ensure that manipulatives promote learning when used with young children. We also describe how Montessori instruction offers a concrete example of the application of these principles in practice, which may, in turn, explain the high levels of mathematics achievement among children who attend Montessori programs during early childhood. The general principles and concrete examples presented in this article should help early childhood programs maximize the benefits of using manipulatives for developmentally appropriate mathematics instruction.

Keywords

manipulatives, Montessori, mathematics, materials, cognitive science

Walk into any early childhood classroom and you are sure to see mathematics manipulatives. Manipulatives are concrete materials (e.g., blocks, tiles) used to demonstrate a mathematics concept or to support the execution of a mathematical procedure. They have become a mainstay of mathematical instruction in America as well as internationally (e.g., Correa, Perry, Sims, Miller, & Fang, 2008; Puchner, Taylor, O'Donnell, & Fick, 2008). In a study of two school districts, the average elementary teacher reported using manipulatives nearly every day (Uribe-Flórez & Wilkins, 2010).

Research examining the advantages of instruction using manipulatives, however, is inconsistent: Some studies find that manipulatives promote learning, whereas others find that they hinder it. A recent meta-analysis of 55 studies that compared instruction with or without manipulatives suggests that manipulatives can benefit learning, but only under certain conditions (Carbonneau, Marley, & Selig, 2013). For instance, differences in the benefits of manipulatives were associated with the content being taught; manipulatives were more advantageous for learning about fractions than for learning arithmetic. The results also indicated that instruction with manipulatives was least effective for children between the ages of 3 and 6 years, with very small and sometimes negative effects. These findings suggest that the efficacy of manipulatives for promoting learning may depend on the conditions under which they are used.

Given the lack of clear evidence supporting the use of manipulatives, should they be used to teach mathematics in early childhood? We believe the answer is yes—*if* careful consideration is given to what research has identified about the conditions under which when manipulatives are likely to promote, rather than hinder, learning. Cognitive science research, in particular, has generated a considerable amount of knowledge that could be useful for improving instruction so that all young children can acquire the mathematics knowledge necessary for success, as described in National Council of Teachers of Mathematics (NCTM) standards and Common Core standards (Laski, Reeves, Ganley, & Mitchell, 2013; NCTM, 2006; National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010; Newcombe et al., 2009; Siegler, 2003). In this article, we discuss the findings from cognitive science relevant to the use of manipulatives in early childhood math instruction, and synthesize them into four principles for maximizing the effective use of mathematics manipulatives.

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To demonstrate how early childhood instruction can reflect these principles, we offer examples from Montessori instruction. Maria Montessori (Montessori & Simmonds, 1917) was among the first educators to develop materials specifically designed to instantiate mathematics concepts. She developed a wide array of materials designed to help children understand concepts, such as place value (Lillard, 2005). Children who attend Montessori programs in early childhood demonstrate high levels of mathematics achievement. Children who were randomly selected to attend a Montessori program scored higher on a standardized math test than children who had not been selected and attended a non-Montessori program (Lillard & Else-Quest, 2006). In fact, close adherence to the Montessori approach seems to promote better math learning: Children who attend high-fidelity Montessori programs are more likely to have higher standardized math scores than those who attend lower fidelity Montessori or traditional early childhood programs (Lillard, 2012). The benefits of the Montessori approach to mathematics learning in early childhood may, at least in part, be due to its effective use of manipulatives.

Four Principles for Maximizing the Effectiveness of Manipulatives

The widespread use of manipulatives is rooted in the idea that young children reason concretely before they do so abstractly. It is important to remember, however, that even though manipulatives are concrete objects, understanding how they represent concepts requires abstract thinking—a manipulative is still just a physical representation of a concept, not the concept itself. Thus, cognitive research about young children's symbolic reasoning and the conditions that facilitate their ability to abstract information from symbols can inform classroom practice that is developmentally appropriate. From our review of the literature, four general principles emerged: (a) use a manipulative consistently, over a long period of time; (b) begin with highly transparent concrete representations and move to more abstract representations over time; (c) avoid manipulatives that resemble everyday objects or have distracting irrelevant features; and (d) explicitly explain the relation between the manipulatives and the math concept. What follows is a description of the research in support of each principle and examples of how Montessori instruction serves as a model of these principles.

Use a Manipulative Consistently, Over a Long Period of Time

What the research says. For manipulatives to be effective, children, particularly young children, need time to make the relation between the concrete materials and the abstract concepts they represent. More than two decades ago, Sowell (1989) conducted one of the first meta-analyses of studies comparing instruction with manipulatives with instruction without it. The strongest conclusion from the data was that the benefit of manipulatives depends on how long children are exposed to them: Exposure to the same manipulative for a school year or more led to moderate effect sizes, whereas instruction with manipulatives over a shorter period of time led to learning levels comparable with those of instruction without manipulatives.

Recent research from cognitive science helps to explain this phenomenon. Young children do not easily interpret the meaning of symbols to use them for problem solving (DeLoache, 2004). For example, children under the age of 5 are unable to make the connection between a scale model of a room and a regular-sized room to locate a hidden toy without receiving explicit guidance from an experimenter (DeLoache, Peralta de Mendoza, & Anderson, 1999). Children become better able to interpret the relation between a symbol and its referent with age, but even older children need cumulative experience with a symbol to use it for sophisticated reasoning (Liben & Myers, 2007). Children are better able to identify the relation between two constructs (or in this case, a concept and a manipulative) when they have multiple opportunities to compare them (Gick & Holyoak, 1983; Son, Smith, & Goldstone, 2011).

Theories of physically distributed learning suggest that using the same or similar manipulatives to repeatedly solve problems leads to a deeper understanding of the relation between the physical material and the abstract concept because it allows for an understanding of the two to co-evolve (Martin, 2009). In other words, using the manipulative helps establish a basic understanding of the math concept that in turn promotes deeper insights into how the material relates to the concept that in turn leads to better understanding of the concept and so on. This iterative cycle, however, is theorized as only being possible when there is consistent prolonged use of the same or similar manipulatives (Martin, 2009).

The Montessori approach. The Montessori approach allows for long-term use of the same or similar manipulatives through both the structure of its programs and the design of the manipulatives. Traditionally, each level of Montessori education encompasses a 3-year mixed age group, so an early childhood classroom includes children aged 3 through 6. This multi-year time frame and the consistency between the early childhood and elementary programs provide extensive opportunities for children to abstract the mathematical concepts represented by the Montessori math manipulatives and to gradually develop more sophisticated knowledge over an extended period of time (Lillard, 2005). Furthermore, materials introduced and used throughout the early childhood level, or slight variations of them, are also used in the elementary grades to explain more advanced concepts.

A second way Montessori instruction allows for children to have extended time with manipulatives is that it uses a limited, but a central, set of math materials to represent number

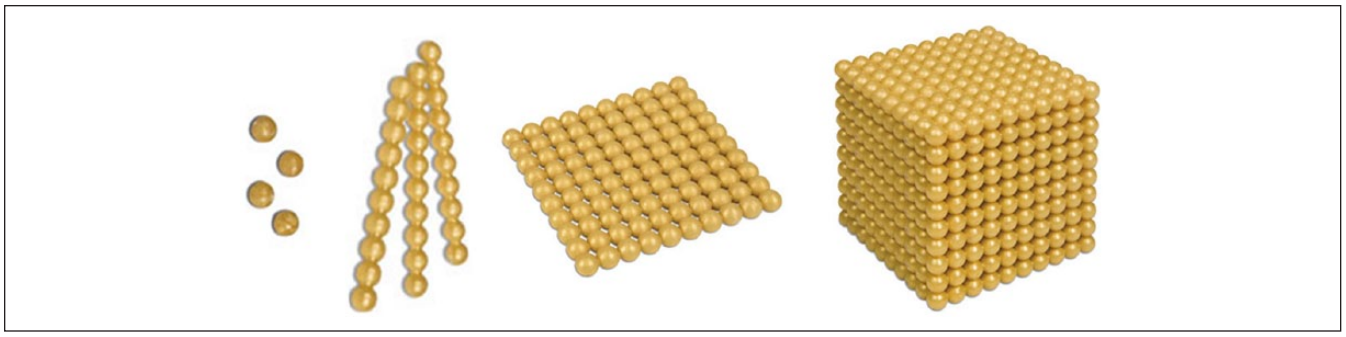

Figure 1. Montessori golden bead materials used for representing number.
Source. Photograph courtesy of Nienhuis Montessori USA, the largest authorized manufacturer of Montessori materials.
Note. From left to right: the unit bead, 10 bar, 100 square, and 1,000 cube.

Table 1. Extended Use of Montessori Golden Bead Manipulative Over Several Years.

	Approximate child age			
	3-4 years	5-6 years	7-8 years	9 to 10+ years
Golden bead material	• Introduction to the decimal system • Exploring different quantities • Association of quantities with numeral cards 1-9,999 • Operations (+/−) with numbers between 1 and 9,999	• Operations (×/÷) with numbers between 1 and 9,999 • Long division • Word problems	• Distributive law • Divisibility	• Squaring a binomial • Square roots for numbers less than 9,999

concepts and operations. One example is the golden bead material (see Figure 1) in which the base-10 number system is represented using identical individual gold colored beads to denote units that are also assembled into bars comprising 10 connected beads, squares that connect beads to form a 10 by 10 square of 100 gold beads, and a cube of 1,000 interconnected beads.

As illustrated in Table 1, the golden beads are used for activities at the early childhood level beginning with the introduction to quantity and numerals and are then used throughout the early elementary years as a basis for explaining the base-10 system and operations, and later to introduce square roots.

Finally, the Montessori approach provides children with multiple opportunities to make connections between a physical representation and the underlying mathematical concept through incorporating the same physical representation in multiple materials (Lillard, 2005). This point can be illustrated through the color coding used to represent place value—for example, green for units, blue for 10s, red for 100s—across various materials. In one instance, this color coding is used for numerals that represent the place value within multi-digit numbers—for example, a child would combine a blue numeral 20 and a green numeral 6 to make the numeral 26 and match it to a set of two 10-bead bars and six unit beads. In another case, small tiles used for counting and arithmetic also follow the same color scheme: 1 tiles are green, 10s are blue, and 100s are red. In another material, a kind of abacus used for representing larger numbers and arithmetic, there is a row of green beads that represent units, a row of blue beads that represent the 10s, and a row of red beads that represent the 100s. Furthermore, as children progress to working with larger numbers, the same color scheme is used to represent the recursive nature of the number system—for example, green is used to represent units of 1,000, blue 10,000s, and red for 100,000s.

Begin With Highly Transparent Concrete Representations and Move to More Abstract Representations Over Time

What the research says. The greater the physical similarity between the manipulative and the concept it represents, the more likely children will be able to understand the relation between the two. Research on the development of symbolic and analogical reasoning provides support for this claim

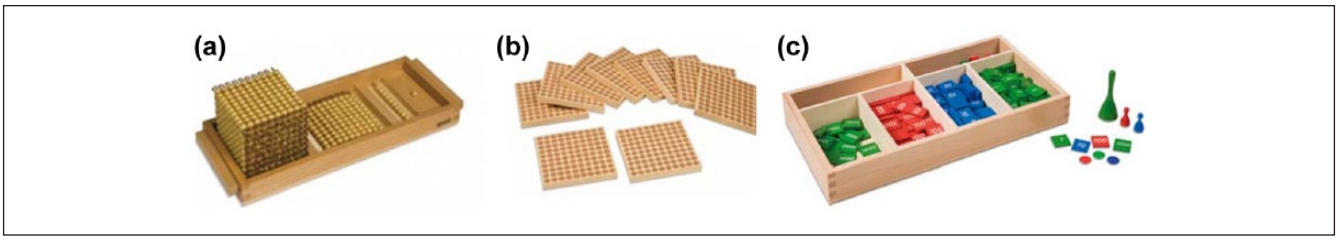

Figure 2. Concreteness fading in the Montessori approach: materials progress from (a) instantiating the individual units in number to (b) wooden squares marked with circles to signify units to (c) wooden tiles with quantity indicated only by numerals.
Source. Photograph courtesy of Nienhuis Montessori USA.

(Chen, 1996; DeLoache, Kolstad, & Anderson, 1991; Gentner & Markman, 1997; Goswami, 1996). For instance, preschoolers are better able to find a hidden toy in a regular-sized room when they are shown its location in a scale model with identical furniture than when they are shown the location in a scale model with generic furniture (DeLoache et al., 1991). Support also comes from research about how board games support learning (Laski & Siegler, 2014; Siegler & Ramani, 2009). A number board game with the numbers 1 to 10 in squares arranged in a line leads to better improvements in preschoolers' understanding of the magnitude of numbers (also known as their mental number line) than a game board with the numbers arranged in a circle (Siegler & Ramani, 2009). It is believed that the linear game board is better because it is a more transparent reflection of increasing numerical magnitude.

Although concrete representations of mathematics concepts are initially important for helping children make the mapping between materials and the concepts they represent, research suggests that instruction should progress to the use of more abstract representations over time. This idea is known as "concreteness fading" (Fyfe, McNeil, Son, & Goldstone, 2014). Carbonneau and colleagues' (2013) meta-analysis of studies testing the effectiveness of manipulatives found that they were more effective for outcomes related to reproducing basic procedures than for outcomes related to transfer (i.e., extending knowledge to new problem types). Recent studies indicate that a systematic fading of concreteness can increase children's ability to transfer knowledge acquired through manipulatives to novel, unfamiliar problems (Fyfe et al., 2014). For instance, children who received instruction about math equivalence problems (e.g., 3 + 4 = 3 + __) solved more transfer problems correctly when instruction progressed from physical objects (i.e., bears on a pan balance) to a worksheet (i.e., illustration of a pan balance) to symbolic equations, compared with children who received instruction in the reverse order or instruction with either only concrete objects or symbolic equations (Fyfe & McNeil, 2009).

The Montessori approach. Montessori materials and instruction exemplify the progression from concrete materials with a high degree of transparency to more abstract materials with less transparency over the 3-year early childhood period (ages 3-6 years). Consider the materials used to support children's understanding of the magnitude of numbers: Instruction on the concept progresses from use of materials that physically represent the composition of numbers through concrete combinations of individual units to tiles with only Arabic numerals that are combined for multi-digit numbers (see Figure 2). Initially the magnitude of numbers is represented using the "golden beads" (mentioned previously)—a collection of beads that contains individual beads, or groups of individual beads organized as 10 bars, 100 squares, or 1,000 cubes. The quantity of individual beads associated with each magnitude (e.g., 10 bar vs. 100 square) directly reflects the magnitude of each number. With time, instruction shifts to increased use of wooden base-10 squares and cubes marked only with circles to signify the collections of beads. The last step in the progression is the use of number tiles of identical size and marked with a 1, 10, 100, or 1,000, which are used without any other physical representation of quantities. Thus, the sequence in which the Montessori materials are introduced is structured to move children to increasingly abstract representations over time.

Avoid Manipulatives That Resemble Everyday Objects or Have Distracting Irrelevant Features

What the research says. Early advocates of manipulatives posited that concrete objects that resemble everyday objects (e.g., teddy bear counters) help children draw on their practical knowledge for understanding concepts (Burns, 1996). Recent research, however, suggests that manipulatives that represent real objects may actually impede learning. In fact, it may be the prevalence of these kinds of manipulatives in early childhood classrooms that explains Carbonneau and colleagues' (2013) finding that instruction with manipulatives was least effective for children between the ages of 3 and 6 years, with very small and sometimes negative effects. More generally, it may be teachers' tendency to allow students to "play" with mathematics manipulatives (Moyer, 2001) that undermines the effectiveness of manipulatives for mathematics learning.

Current research suggests that manipulatives that are as basic as possible (e.g., same colored cubes vs. teddy bear counters) without irrelevant perceptual features or references to real-world objects seem to promote the greatest learning. For example, McNeil, Uttal, Jarvin, and Sternberg (2009) found that children who solved word problems involving money using highly realistic dollar bills and coins made a greater number of errors than those who solved the same problems using more basic representations of money, specifically white pieces of paper with only numbers on them.

Research about young children's symbolic reasoning, specifically evidenced in the dual representation theory, provides an explanation for why manipulatives without irrelevant features are more effective for learning (see McNeil & Uttal, 2009; Uttal, O'Doherty, Newland, Hand, & DeLoache, 2009, for reviews). From the perspective of the dual representation theory, manipulatives can be thought of in two different ways: (a) as objects in their own right and (b) as symbols for mathematics concepts. When the manipulative itself is interesting to play with (e.g., acting out a story with the teddy bear counters or pretending to eat plastic fruit) or elicits ideas irrelevant to the mathematics (e.g., playing with stuffed animals), it distracts and prevents the child from making the relation between the manipulative and the mathematics concept it is meant to represent. However, when the manipulative is basic—stripped of irrelevant perceptual features or attributes—then it helps children direct all of their attention to thinking about its relation with the mathematics concept it represents.

The Montessori approach. Montessori math manipulatives are basic representations of mathematical entities that do not resemble real objects or possess irrelevant perceptual features. For example, the materials described above (and illustrated in Figures 1 and 2) used for representing number quantity and counting activities have no connection with everyday objects. The beads are all the same color and the only differences between them are the quantity they represent (e.g., 10 bar vs. 100 square). Another example is a set of 10 wooden rods illustrated in Figure 3 that range in length from 1 to 10 segments (each 1 dm) with segments alternately painted red and blue that are used to teach the magnitude and order of numbers between 1 and 10. The rods instantiate the quantity of units associated with each number (i.e., the number of individual segments in a given rod), the overall magnitude of a number (i.e., the length of a rod), and the relative magnitude of numbers (i.e., the "two" rod has fewer units and is shorter than the "eight" rod). Also, when children order the rods, they see a concrete representation of the successor rule—each subsequent number is exactly one more unit than the previous number. Because the rods are all perceptually identical (i.e., same color, texture, thickness), except for the relevant attributes (i.e., number of segments and length) children's attention is drawn to the relevant features of the rods and there are no irrelevant features to distract them. Thus, the simplicity of Montessori materials is that though they are superficially less interesting or appealing than more broadly used manipulatives, they are designed in ways that are more likely to focus children's attention on the attributes that represent the mathematical concept and increase learning.

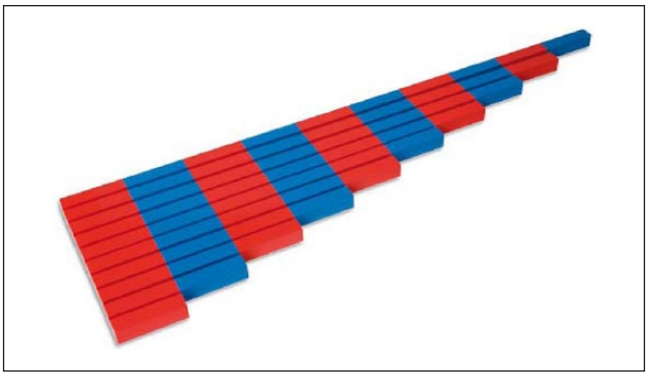

Figure 3. Set of Montessori rods used to represent increasing number size.
Source. Photograph courtesy of Nienhuis Montessori USA.

Explicitly Explain the Relation Between the Manipulatives and the Math Concept

What the research says. Finally, with even the best designed manipulatives, it is unreasonable to expect young children to make the relation between the concrete material and the mathematics concept it represents without explicit guidance (Ball, 1992; McNeil & Jarvin, 2007). Studies of children's symbolic reasoning consistently find that children under the age of 5 have trouble abstracting the meaning of a symbol without instruction (e.g., DeLoache et al., 1999). This research suggests that explicit statements about how the material represents the mathematical procedure or concept helps direct children's attention to the relevant features of the materials. Directing attention may, in turn, promote learning because it allows children's limited cognitive resources to focus on the mathematics rather than on trying to abstract the relation between the material and the mathematics concept (Kirschner, Sweller, & Clark, 2006). Consistent with these findings from cognitive research, Deborah Ball (1992), an expert in mathematics education, argued strongly against a constructivist view of manipulatives and the idea that children can independently develop an understanding of mathematics concepts by interacting with concrete materials: "Although kinesthetic experiences can enhance perception and thinking, understanding does not travel through the fingertips and up the arm" (p. 47).

Indeed, differences in the extent to which teachers provide guidance when using manipulatives or other models are attributed to differences in student learning and mathematics achievement (e.g., Boulton-Lewis & Tait, 1994; Fuson & Briars, 1990; Hiebert & Wearne, 1992). For example,

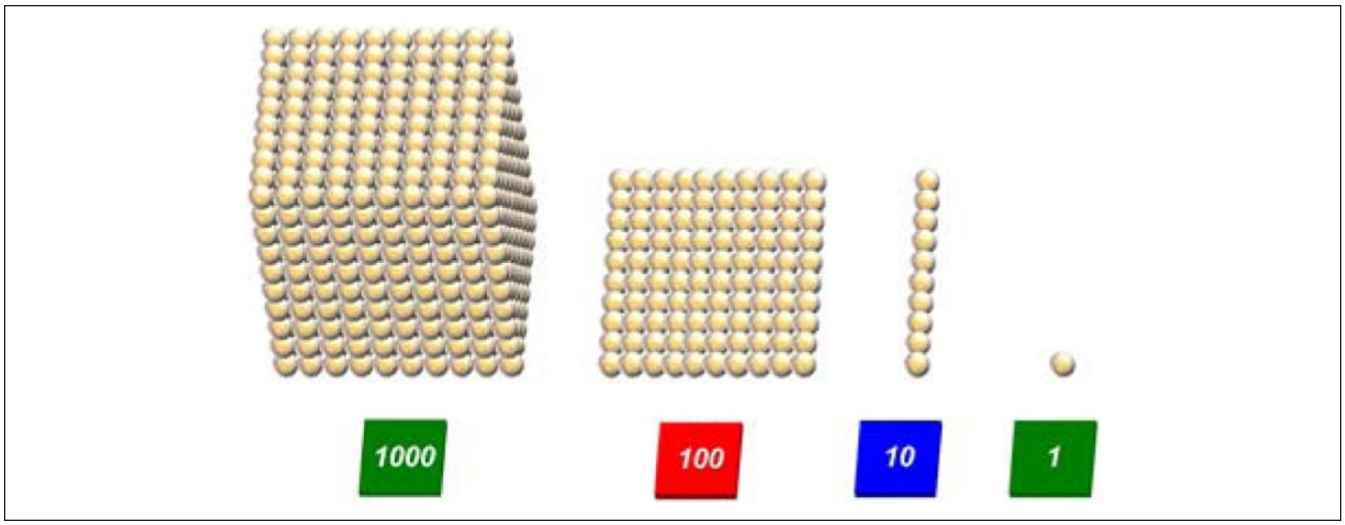

Figure 4. Physical alignment of materials used to help children connect a more abstract representation (i.e., numeral tiles) to earlier used and practiced concrete representations (i.e., bead materials).

Richland, Zur, and Holyoak (2007) found that teachers in Hong Kong and Japan were more likely than U.S. teachers to provide guidance when presenting analogies in mathematics and that this may contribute to the higher performance of students from these nations on cross-national assessments of mathematics achievement. Carbonneau and colleagues' (2013) meta-analysis of studies testing the effectiveness of manipulatives similarly found that studies in which the use of manipulatives was accompanied by high levels of instructional guidance led to greater effect sizes than studies in which low levels of guidance were used.

The guidance provided can be either verbal or non-verbal. In fact, gestures have been found to be a particularly effective instructional tool even when they provide information different from the strategy explained verbally (Singer & Goldin-Meadow, 2005). More specifically, "linking gestures" are believed to play an important role in directing children's attention to the connection between two representations (Alibali & Nathan, 2007; Richland, 2008). A teacher, for instance, who points to a fulcrum of a pan balance and then to an equal sign is using gesture to help children understand the connection between the concrete and symbolic representation of equality (Alibali & Nathan, 2007).

The Montessori approach. In Montessori instruction, early childhood teachers use both gesture and language to help children see the relation between mathematics materials and the concepts they are meant to represent by drawing children's attention to the relevant features of the materials. For example, when children are first introduced to the golden bead materials (see Figure 1), the teacher explicitly points out to the child the value of the beads; the teacher places a single unit bead in front of the child and says, "This is a unit." Later, when the golden bead materials are used to teach children about number and counting, the teacher points as she counts each bead, helping them to make the connection between the quantity and the number words. Similarly as the materials begin to be used to explain place-value concepts and the carry-over procedure, language is used in conjunction with gesture to facilitate children's understanding of the mathematics concept being demonstrated. For example, a teacher would count out 9 unit beads, then, before a 10th bead is added, would ask the child, "Nine units and one more unit would be how many?" As the child says, "10," the teacher replaces the nine unit beads with a single 10 bar, points to the 10 bar, and says, "One more would be ten or one ten."

Montessori instruction also provides guidance to help children see the connection between increasingly abstract sets of materials. For example, when the colored number tiles are first introduced to children, they are explicitly connected to the more concrete representation of numerical quantity used earlier, the golden beads. First, the teacher reminds children of the value of the bead materials (unit bead, 10 bar, etc.). Then, as illustrated in Figure 4, the number tiles are placed directly in front of the bead materials with the same magnitude as the teacher names the numeral on the tile. This kind of physical alignment, accompanied by verbal explanation, is consistent with the kind of instruction that has been found to help children notice how two representations are connected (Richland et al., 2007).

Conclusion

Despite the widespread use of manipulatives in early childhood mathematics instruction, research examining the efficacy of manipulatives for mathematics instruction is inconsistent. In fact, a recent large meta-analysis of studies that compared instruction with or without manipulatives

indicated that instruction with manipulatives was least effective for children between the ages of 3 and 6 years, with very small and sometimes negative effects (Carbonneau et al., 2013). Thus, it is imperative that early childhood educators think carefully about ways to effectively use mathematics manipulatives for learning and use research to guide them.

Over the past two decades, there has been increased recognition that cognitive science research can and should inform education (e.g., Bransford, Brown, & Cocking, 2000; Newcombe et al., 2009; Siegler, 2003). Indeed, the field has generated a considerable amount of knowledge that could be useful for improving instruction so that all young children acquire foundational mathematics knowledge (Laski et al., 2013; Siegler, 2003). In this article, we reviewed the findings most relevant to the use of manipulatives in early childhood math instruction, identifying four general principles: (a) use a manipulative consistently, over a long period of time; (b) begin with highly transparent concrete representations and move to more abstract representations over time; (c) avoid manipulatives that resemble everyday objects or have distracting irrelevant features; and (d) explicitly explain the relation between the manipulatives and the math concept.

Cognitive science research suggests that instruction that follows these principles when using manipulatives is likely to lead to greater mathematics learning than instruction that does not. Indeed, the Montessori approach to mathematics instruction in early childhood uses manipulatives in a manner consistent with these principles, and children who attend Montessori programs in early childhood demonstrate high levels of mathematics achievement (Lillard, 2012; Lillard & Else-Quest, 2006). The Montessori examples provided in this article, however, illustrate just one approach to how these principles can be translated to practice.

Any early childhood program can apply the principles and, in most cases, through fairly minor changes in practice. For example, to ensure that the same or similar manipulatives are used over a long period of time and that instruction progresses from concrete to abstract representations, programs could allow for administrators and teachers across various age groups and grade levels to collaboratively select and sequence which manipulatives will be used at each level. To ensure that the manipulatives used in instruction have few distracting features, teachers could minimize or eliminate the use of theme-based manipulatives (e.g., bug or teddy bear counters) and move instead toward using one or two general manipulatives (e.g., Cuisenaire rods, counting chips) for mathematics activities. Simple modifications to instruction based on the principles presented here are likely to increase the effective use of manipulatives in mathematics instruction and strengthen children's problem solving, critical thinking, and learning outcomes in mathematics.

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